NAG Toolbox for MATLAB

f08wb

1 Purpose

f08wb computes for a pair of n by n real nonsymmetric matrices (A, B) the generalized eigenvalues and, optionally, the left and/or right generalized eigenvectors using the QZ algorithm.

Optionally it also computes a balancing transformation to improve the conditioning of the eigenvalues and eigenvectors, reciprocal condition numbers for the eigenvalues, and reciprocal condition numbers for the right eigenvectors.

2 Syntax

```
[a, b, alphar, alphai, beta, vl, vr, ilo, ihi, lscale, rscale, abnrm,
bbnrm, rconde, rcondv, info] = f08wb(balanc, jobvl, jobvr, sense, a, b,
'n', n)
```

3 Description

A generalized eigenvalue for a pair of matrices (A, B) is a scalar λ or a ratio $\alpha/\beta = \lambda$, such that $A - \lambda B$ is singular. It is usually represented as the pair (α, β), as there is a reasonable interpretation for $\beta = 0$, and even for both being zero.

The right eigenvector v_j corresponding to the eigenvalue λ_j of (A, B) satisfies

$$Av_j = \lambda_j Bv_j.$$

The left eigenvector u_j corresponding to the eigenvalue λ_j of (A, B) satisfies

$$u_j^{\mathrm{H}} A = \lambda_j u_j^{\mathrm{H}} B.$$

where u_j^{H} is the conjugate-transpose of u_j.

All the eigenvalues and, if required, all the eigenvectors of the generalized eigenproblem $Ax = \lambda Bx$, where A and B are real, square matrices, are determined using the QZ algorithm. The QZ algorithm consists of four stages:

(i) A is reduced to upper Hessenberg form and at the same time B is reduced to upper triangular form.

(ii) A is further reduced to quasi-triangular form while the triangular form of B is maintained. This is the real generalized Schur form of the pair (A, B).

(iii) The quasi-triangular form of A is reduced to triangular form and the eigenvalues extracted. This function does not actually produce the eigenvalues λ_j, but instead returns α_j and β_j such that

$$\lambda_j = \alpha_j/\beta_j, \qquad j = 1, 2, \ldots, n.$$

The division by β_j becomes your responsibility, since β_j may be zero, indicating an infinite eigenvalue. Pairs of complex eigenvalues occur with α_j/β_j and α_{j+1}/β_{j+1} complex conjugates, even though α_j and α_{j+1} are not conjugate.

(iv) If the eigenvectors are required they are obtained from the triangular matrices and then transformed back into the original co-ordinate system.

For details of the balancing option, see Section 3 of the document for f08wh.

4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D 1999 *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia URL: http://www.netlib.org/lapack/lug

Golub G H and Van Loan C F 1996 *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Wilkinson J H 1979 Kronecker's canonical form and the *QZ* algorithm *Linear Algebra Appl.* **28** 285–303

5 Parameters

5.1 Compulsory Input Parameters

1: **balanc – string**

Specifies the balance option to be performed.

balanc = 'N'

Do not diagonally scale or permute.

balanc = 'P'

Permute only.

balanc = 'S'

Scale only.

balanc = 'B'

Both permute and scale.

Computed reciprocal condition numbers will be for the matrices after permuting and/or balancing. Permuting does not change condition numbers (in exact arithmetic), but balancing does. In the absence of other information, **balanc** = 'B' is recommended.

Constraint: **balanc** = 'N', 'P', 'S' or 'B'.

2: **jobvl – string**

If **jobvl** = 'N', do not compute the left generalized eigenvectors.

If **jobvl** = 'V', compute the left generalized eigenvectors.

Constraint: **jobvl** = 'N' or 'V'.

3: **jobvr – string**

If **jobvr** = 'N', do not compute the right generalized eigenvectors.

If **jobvr** = 'V', compute the right generalized eigenvectors.

Constraint: **jobvr** = 'N' or 'V'.

4: **sense – string**

Determines which reciprocal condition numbers are computed.

sense = 'N'

None are computed.

sense = 'E'

Computed for eigenvalues only.

sense = 'V'

Computed for eigenvectors only.

sense = 'B'

Computed for eigenvalues and eigenvectors.

Constraint: **sense** = 'N', 'E', 'V' or 'B'.

5: **a(lda,$*$) – double array**

The first dimension of the array **a** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

The matrix A in the pair (A,B).

6: **b(ldb,$*$) – double array**

The first dimension of the array **b** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

The matrix B in the pair (A,B).

5.2 Optional Input Parameters

1: **n – int32 scalar**

Default: The first dimension of the arrays **a**, **b** and the second dimension of the arrays **a**, **b**. (An error is raised if these dimensions are not equal.)

n, the order of the matrices A and B.

Constraint: $\mathbf{n} \geq 0$.

5.3 Input Parameters Omitted from the MATLAB Interface

lda, ldb, ldvl, ldvr, work, lwork, iwork, bwork

5.4 Output Parameters

1: **a(lda,$*$) – double array**

The first dimension of the array **a** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

a has been overwritten. If **jobvl** = 'V' or **jobvr** = 'V' or both, then A contains the first part of the real Schur form of the ‘balanced’ versions of the input A and B.

2: **b(ldb,$*$) – double array**

The first dimension of the array **b** must be at least $\max(1,\mathbf{n})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

b has been overwritten.

3: **alphar($*$) – double array**

Note: the dimension of the array **alphar** must be at least $\max(1,\mathbf{n})$.

The element **alphar**(j) contains the real part of α_j.

4: **alphai($*$) – double array**

Note: the dimension of the array **alphai** must be at least $\max(1,\mathbf{n})$.

The element **alphai**(j) contains the imaginary part of α_j.

5: **beta($*$) – double array**

Note: the dimension of the array **beta** must be at least $\max(1,\mathbf{n})$.

$(\mathbf{alphar}(j) + \mathbf{alphai}(j) \times i)/\mathbf{beta}(j)$, for $j = 1,\ldots,\mathbf{n}$, will be the generalized eigenvalues.

If **alphai**(j) is zero, then the jth eigenvalue is real; if positive, then the jth and $(j+1)$st eigenvalues are a complex conjugate pair, with **alphai**$(j+1)$ negative.

Note: the quotients **alphar**$(j)/$**beta**(j) and **alphai**$(j)/$**beta**(j) may easily overflow or underflow, and **beta**(j) may even be zero. Thus, you should avoid naively computing the ratio α_j/β_j. However, $\max|\alpha_j|$ will always be less than and usually comparable with $\|\mathbf{a}\|_2$ in magnitude, and $\max|\beta_j|$ will always be less than and usually comparable with $\|\mathbf{b}\|_2$.

6: **vl(ldvl,$*$) – double array**

The first dimension, **ldvl**, of the array **vl** must satisfy

if **jobvl** $=$ 'V', **ldvl** $\geq \max(1,\mathbf{n})$;
ldvl ≥ 1 otherwise.

The second dimension of the array must be at least $\max(1,\mathbf{n})$ if **jobvl** $=$ 'V', and at least 1 otherwise

If **jobvl** $=$ 'V', the left eigenvectors u_j are stored one after another in the columns of **vl**, in the same order as the corresponding eigenvalues.

If the jth eigenvalue is real, then $u_j = \mathbf{vl}(:,j)$, the jth column of **vl**.

If the jth and $(j+1)$th eigenvalues form a complex conjugate pair, then $u_j = \mathbf{vl}(:,j) + i \times \mathbf{vl}(:,j+1)$ and $u(j+1) = \mathbf{vl}(:,j) - i \times \mathbf{vl}(:,j+1)$. Each eigenvector will be scaled so the largest component has $|\text{real part}| + |\text{imag. part}| = 1$.

If **jobvl** $=$ 'N', **vl** is not referenced.

7: **vr(ldvr,$*$) – double array**

The first dimension, **ldvr**, of the array **vr** must satisfy

if **jobvr** $=$ 'V', **ldvr** $\geq \max(1,\mathbf{n})$;
ldvr ≥ 1 otherwise.

The second dimension of the array must be at least $\max(1,\mathbf{n})$ if **jobvr** $=$ 'V', and at least 1 otherwise

If **jobvr** $=$ 'V', the right eigenvectors v_j are stored one after another in the columns of **vr**, in the same order as their eigenvalues.

If the jth eigenvalue is real, then $v(j) = \mathbf{vr}(:,j)$, the jth column of VR.

If the jth and $(j+1)$th eigenvalues form a complex conjugate pair, then $v_j = \mathbf{vr}(:,j) + i \times \mathbf{vr}(:,j+1)$ and $v_{j+1} = \mathbf{vr}(:,j) - i \times \mathbf{vr}(:,j+1)$.

Each eigenvector will be scaled so the largest component has $|\text{real part}| + |\text{imag. part}| = 1$.

If **jobvr** $=$ 'N', **vr** is not referenced.

8: **ilo – int32 scalar**
9: **ihi – int32 scalar**

ilo and **ihi** are integer values such that $\mathbf{a}(i,j) = 0$ and $\mathbf{b}(i,j) = 0$ if $i > j$ and $j = 1,\dots,\mathbf{ilo} - 1$ or $i = \mathbf{ihi} + 1,\dots,\mathbf{n}$.

If **balanc** $=$ 'N' or 'S', **ilo** $= 1$ and **ihi** $= \mathbf{n}$.

10: **lscale($*$) – double array**

Note: the dimension of the array **lscale** must be at least $\max(1,\mathbf{n})$.

Details of the permutations and scaling factors applied to the left side of A and B.

If pl_j is the index of the row interchanged with row j, and dl_j is the scaling factor applied to row j, then:

lscale$(j) = pl_j$, for $j = 1, \ldots,$ **ilo** $- 1$;

lscale $= dl_j$, for $j =$ **ilo**$, \ldots,$ **ihi**;

lscale $= pl_j$, for $j =$ **ihi** $+ 1, \ldots,$ **n**.

The order in which the interchanges are made is **n** to **ihi** $+ 1$, then 1 to **ilo** $- 1$.

11: **rscale**$(*)$ **– double array**

Note: the dimension of the array **rscale** must be at least $\max(1, \mathbf{n})$.

Details of the permutations and scaling factors applied to the right side of A and B.

If pr_j is the index of the column interchanged with column j, and dr_j is the scaling factor applied to column j, then:

rscale$(j) = pr_j$, for $j = 1, \ldots,$ **ilo** $- 1$;
if **rscale** $= dr_j$, for $j =$ **ilo**$, \ldots,$ **ihi**;
if **rscale** $= pr_j$, for $j =$ **ihi** $+ 1, \ldots,$ **n**.

The order in which the interchanges are made is **n** to **ihi** $+ 1$, then 1 to **ilo** $- 1$.

12: **abnrm – double scalar**

The 1-norm of the balanced matrix A.

13: **bbnrm – double scalar**

The 1-norm of the balanced matrix B.

14: **rconde**$(*)$ **– double array**

Note: the dimension of the array **rconde** must be at least $\max(1, \mathbf{n})$.

If **sense** = 'E' or 'B', the reciprocal condition numbers of the eigenvalues, stored in consecutive elements of the array. For a complex conjugate pair of eigenvalues two consecutive elements of **rconde** are set to the same value. Thus **rconde**(j), **rcondv**(j), and the jth columns of **vl** and **vr** all correspond to the jth eigenpair.

If **sense** = 'V', **rconde** is not referenced.

15: **rcondv**$(*)$ **– double array**

Note: the dimension of the array **rcondv** must be at least $\max(1, \mathbf{n})$.

If **sense** = 'V' or 'B', the estimated reciprocal condition numbers of the eigenvectors, stored in consecutive elements of the array. For a complex eigenvector two consecutive elements of **rcondv** are set to the same value.

If **sense** = 'E', **rcondv** is not referenced.

16: **info – int32 scalar**

info $= 0$ unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

info $= -i$

If **info** $= -i$, parameter i had an illegal value on entry. The parameters are numbered as follows:

1: **balanc**, 2: **jobvl**, 3: **jobvr**, 4: **sense**, 5: **n**, 6: **a**, 7: **lda**, 8: **b**, 9: **ldb**, 10: **alphar**, 11: **alphai**, 12: **beta**, 13: **vl**, 14: **ldvl**, 15: **vr**, 16: **ldvr**, 17: **ilo**, 18: **ihi**, 19: **lscale**, 20: **rscale**, 21: **abnrm**, 22: **bbnrm**, 23: **rconde**, 24: **rcondv**, 25: **work**, 26: **lwork**, 27: **iwork**, 28: **bwork**, 29: **info**.

It is possible that **info** refers to a parameter that is omitted from the MATLAB interface. This usually indicates that an error in one of the other input parameters has caused an incorrect value to be inferred.

info $= 1$ to N

The *QZ* iteration failed. No eigenvectors have been calculated, but **alphar**(j), **alphai**(j), and **beta**(j) should be correct for $j =$ **info** $+ 1, \ldots,$ **n**.

info $= N + 1$

Unexpected error returned from f08xe.

info $= N + 2$

Error returned from f08yk.

7 Accuracy

The computed eigenvalues and eigenvectors are exact for a nearby matrices $(A + E)$ and $(B + F)$, where

$$\|(E, F)\|_F = O(\epsilon)\|(A, B)\|_F,$$

and ϵ is the ***machine precision***.

An approximate error bound on the chordal distance between the ith computed generalized eigenvalue w and the corresponding exact eigenvalue λ is

$$\epsilon \times \|\mathbf{abnrm}, \mathbf{bbnrm}\|_2 / \mathbf{rconde}(i).$$

An approximate error bound for the angle between the ith computed eigenvector **vl**(i) or **vr**(i) is given by

$$\epsilon \times \|\mathbf{abnrm}, \mathbf{bbnrm}\|_2 / \mathbf{rcondv}(i).$$

For further explanation of the reciprocal condition numbers **rconde** and **rcondv**, see Section 4.11 of Anderson *et al.* 1999.

Note: interpretation of results obtained with the *QZ* algorithm often requires a clear understanding of the effects of small changes in the original data. These effects are reviewed in Wilkinson 1979, in relation to the significance of small values of α_j and β_j. It should be noted that if α_j and β_j are **both** small for any j, it may be that no reliance can be placed on **any** of the computed eigenvalues $\lambda_i = \alpha_i / \beta_i$. You are recommended to study Wilkinson 1979 and, if in difficulty, to seek expert advice on determining the sensitivity of the eigenvalues to perturbations in the data.

8 Further Comments

The total number of floating-point operations is proportional to n^3.

The complex analogue of this function is f08wp.

9 Example

```
balanc = 'Balance';
jobvl = 'No vectors (left)';
jobvr = 'Vectors (right)';
sense = 'Both reciprocal condition numbers';
a = [3.9, 12.5, -34.5, -0.5;
     4.3, 21.5, -47.5, 7.5;
     4.3, 21.5, -43.5, 3.5;
     4.4, 26, -46, 6];
b = [1, 2, -3, 1;
     1, 3, -5, 4;
     1, 3, -4, 3;
     1, 3, -4, 4];
```

```
[aOut, bOut, alphar, alphai, beta, vl, vr, ilo, ihi, lscale, rscale, ...
 abnrm, bbnrm, rconde, rcondv, info] = ...
    f08wb(balanc, jobvl, jobvr, sense, a, b)
```

```
aOut =
    3.2224   -9.6500    6.6759    9.2824
         0    0.2275    0.7369    4.4114
         0   -0.2267    0.2253    2.4373
         0         0         0    2.8425
bOut =
    1.6112   -1.0652    0.7761    6.1473
         0    0.1849         0    1.8929
         0         0    0.0472    0.8504
         0         0         0    0.7106
alphar =
    3.2224
    0.3880
    0.2026
    2.8425
alphai =
         0
    0.5173
   -0.2701
         0
beta =
    1.6112
    0.1293
    0.0675
    0.7106
vl =
     0
vr =
   -1.0000   -0.4255   -0.5745   -1.0000
   -0.0057   -0.0851   -0.1149   -0.0111
   -0.0629   -0.1430   -0.0009    0.0333
   -0.0629   -0.1430   -0.0009   -0.1556
ilo =
           1
ihi =
           4
lscale =
     1
     1
     1
     1
rscale =
    1.0000
    0.1000
    0.1000
    1.0000
abnrm =
   17.5000
bbnrm =
    12
rconde =
    0.0952
    0.1652
    0.1652
    0.5141
rcondv =
    0.1254
    0.0381
    0.0381
    0.0707
info =
           0
```